Notes for AA214, Chapter 3

T. H. Pulliam
Stanford University

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Modified Wave Number Analysis

1. Arbitrary periodic functions can be decomposed into their Fourier components, which are in the form $e^{i\kappa x}$, where κ is the wavenumber. For a general κ

$$u(x) = c_\kappa e^{i\kappa x}$$

2. The exact derivative in x

$$\frac{\partial\, u(x)}{\partial x} = i\kappa c_\kappa e^{i\kappa x} = i\kappa\, u(x)$$

3. How will a finite-difference operator δ_x approximate the derivative of $u_j = c_\kappa e^{i\kappa x_j}$, $x_j = j\Delta x$

4. By definition we have ($i\kappa^*$ is the modified wave number)

$$\delta_x u_j = i\kappa^* c_\kappa e^{i\kappa x} = i\kappa^* u_j$$

5. The particular form of $i\kappa^*$ depends on the choice of δ_x

Modified Wave Number - Central Differencing

1. Central Difference:

$$\delta_x^c\, u_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x}$$

2. Using $u_j = e^{i\kappa j \Delta x}$ we have

$$\delta_x^c u_j = \frac{e^{i\kappa (j+1)\Delta x} - e^{i\kappa (j-1)\Delta x}}{2\Delta x} = \frac{e^{\,i\kappa\Delta x} - e^{-\,i\kappa\Delta x}}{2\Delta x} e^{i\kappa j\Delta x} = \frac{e^{\,i\kappa\Delta x} - e^{-\,i\kappa\Delta x}}{2\Delta x} u_j = i\kappa_c^* u_j$$

3. Using the definition of the complex exponential $e^{\,i\kappa\Delta x} = cos(\kappa\Delta x) + isin(\kappa\Delta x)$ we have

$$i\kappa_c^* = i\,\frac{sin(\kappa\Delta x)}{\Delta x}$$

4. The modified wave number $i\kappa^*$ is an approximation to $i\kappa$ the "exact" wave number.

5. For δ_x^c, using the infinite series expansion of $sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$$i\kappa_c^* = i\,\frac{sin(\kappa\Delta x)}{\Delta x} = \frac{i}{\Delta x}\left[(\kappa\Delta x) - \frac{(\kappa\Delta x)^3}{6} + O(\Delta x^5)\right] = i\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\right]$$

6. Therefore $i\kappa_c^* = i\kappa - i\kappa\frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4) = i\kappa + O(\Delta x^2)$ a second order approximation.

Modified Wave Number - 1^{st} Order Backward Differencing

1. Backward Difference:
$$\delta_x^b\, u_j = \frac{u_j - u_{j-1}}{\Delta x}$$

2. Using $u_j = e^{i\kappa j\Delta x}$ we have
$$\delta_x^b u_j = \frac{e^{i\kappa j\Delta x} - e^{i\kappa(j-1)\Delta x}}{\Delta x} = \frac{1 - e^{-i\kappa\Delta x}}{\Delta x} e^{i\kappa j\Delta x} = \frac{1 - e^{-i\kappa\Delta x}}{\Delta x} u_j = i\kappa_b^* u_j$$

3. Expanding in sin and cos
$$i\kappa_b^* = \frac{1 - cos(\kappa\Delta x) + isin(\kappa\Delta x)}{\Delta x}$$

4. For δ_x^b, using the infinite series expansion of $sin(x)$ and $cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$
$$\begin{aligned} i\kappa_b^* &= \frac{1}{\Delta x}\left[\left[\frac{(\kappa\Delta x)^2}{2} + O(\Delta x^4)\right] + i\left[(\kappa\Delta x) - \frac{(\kappa\Delta x)^3}{6} + O(\Delta x^5)\right]\right] \\ &= \frac{\kappa^2\Delta x}{2} + O(\Delta x^3) + i\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\right] \end{aligned}$$

5. Therefore $i\kappa_b^* = i\kappa + O(\Delta x)$ a first order approximation.

Solution to the Discrete PDE

1. The discrete PDE is
$$\frac{\partial u(t)_j}{\partial t} + a\delta_x u(t)_j = 0$$

2. Using seperation of variables: $u(t)_j = e^{i\kappa j\Delta x} f(t)$ and applying the general result $\delta_x u_j = i\kappa^* u_j$
$$\frac{\partial e^{i\kappa j\Delta x} f(t)}{\partial t} + ai\kappa^* e^{i\kappa j\Delta x} f(t) = 0$$

3. The ODE for $f(t)$ is $\frac{\partial f(t)}{\partial t} + af(t)i\kappa^* = 0$ with solution $f(t) = f(0)e^{-ai\kappa^*\, t}$ giving
$$u(t)_j = c_\kappa e^{i\kappa j\Delta x} e^{-ai\kappa^*\, t}, \quad c_\kappa = f(0)$$

4. Comparing this discrete solution with the continuous solution
$$u(x,t) = c_\kappa e^{i\kappa x} e^{-ai\kappa\, t}$$
we can see how the choice of δ_x affects the phase and amplitude of the computed solution.

Effect of Modified Wave

1. For central differencing $i\kappa_c^* = i\kappa - i\kappa\frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)$, plugging it into the discrete solution gives

$$u(t)_j = c_\kappa e^{i\kappa j\Delta x} e^{-ai\kappa - i\kappa\frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\ t} = c_\kappa e^{i\kappa j\Delta x} e^{-ai\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6} + O(\Delta x^4)\right]\ t}$$

2. Dropping the $O(\Delta x^4)$ term and defining $a^* = a\left[1 - \frac{(\kappa\Delta x)^2}{6}\right]$ the modified wave speed

$$u(t)_j = c_\kappa e^{i\kappa j\Delta x} e^{-a^* i\kappa\ t}$$

   as the discrete solution using central differencing.

3. This shows that each wave slows down by $\frac{(\kappa\Delta x)^2}{6}$ which is a function of κ.

4. Following the same reasoning for the backward differencing $i\kappa_b^*$

$$u(t)_j = c_\kappa e^{i\kappa j\Delta x} e^{-a\left[\frac{\kappa^2\Delta x}{2} + i\kappa\left[1 - \frac{(\kappa\Delta x)^2}{6}\right]\right]\ t}$$

   slowing down the waves and also damping them by $\frac{\kappa^2\Delta x}{2}$

5. We can characterize the error by plotting $\kappa^*\Delta x$ against $\kappa\Delta x$